

Determinantal representations of W-weighted Drazin inverse solutions of some quaternion matrix equations

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Abstract: By using determinantal representations of the W-weighted Drazin inverse previously obtained by the author within the framework of the theory of the column-row determinants, we get explicit formulas for determinantal representations of the W-weighted Drazin inverse solutions (analogs of Cramer's rule) of the quaternion matrix equations $\mathbf{WAWX} = \mathbf{D}$, $\mathbf{XWAW} = \mathbf{D}$, and $\mathbf{W}_1\mathbf{AW}_1\mathbf{XW}_2\mathbf{BW}_2 = \mathbf{D}$.

Keywords: W-weighted Drazin inverse, Quaternion matrix, Cramer rule

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1 Introduction

Throughout the paper, we denote the real number field by \mathbb{R} , the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \{a_0 + a_1i + a_2j + a_3k \mid i^2 = j^2 = k^2 = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

by $\mathbb{H}^{m \times n}$, and by $\mathbb{H}_r^{m \times n}$ the set of all $m \times n$ matrices over \mathbb{H} with a rank r . Let $M(n, \mathbb{H})$ be the ring of $n \times n$ quaternion matrices. For $\mathbf{A} \in \mathbb{H}^{n \times m}$, the symbols \mathbf{A}^* stands for the conjugate transpose (Hermitian adjoint) matrix of \mathbf{A} . The matrix $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n \times n}$ is Hermitian if $\mathbf{A}^* = \mathbf{A}$.

In the past, researches into the quaternion skew field had more a theoretical importance, but now a growing number of investigations give wide practical applications of quaternions. In particular through their attitude orientation, the quaternions arise in various fields such as quaternionic quantum theory [1], fluid mechanics and particle dynamics [2, 3], computer graphics [4], aircraft orientation [5], robotic systems [6], life science [7, 8] and etc.

Research on quaternion matrix equations and generalized inverses, which are usefulness tools used to solve matrix equations, has been actively ongoing for more recent years. We mention only some recent papers. Yuan, Wang and Duan [9] derived solutions of the quaternion matrix equation $AX = B$ and their applications in color image restoration. Wang and Yu [10] studied extreme ranks of real matrices in solution of the quaternion matrix equation $AXB = C$. Yuan, Liao and Lei [11] obtained the expressions of least squares Hermitian solution

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with minimum norm of the quaternion matrix equation $(AXB, CXD) = (E, F)$. Feng and Cheng [12] gave a clear description of the solution set to the quaternion matrix equation $AX - \bar{X}B = 0$. Jiang and Wei [13] derived the explicit solution of the quaternion matrix equation $X - A\bar{X}B = C$. Song, Chen and Wang [14] obtained the expressions of the explicit solutions of quaternion matrix equations $XF - AX = BY$ and $XF - A\bar{X} = BY$. Yuan and Wang [15] gave the expressions of the least squares η -Hermitian solution with the least norm of the quaternion matrix equation $AXB + CXD = E$. Zhang, Wei, Lia and Zhao derived [16] the expressions of the minimal norm least squares solution, the pure imaginary least squares solution, and the real least squares solution for the quaternion matrix equation $AX = B$.

The definitions of the generalized inverse matrices have been extended to quaternion matrices as follows.

The Moore-Penrose inverse of $\mathbf{A} \in \mathbb{H}^{m \times n}$, denoted by \mathbf{A}^\dagger , is the unique matrix $\mathbf{X} \in \mathbb{H}^{n \times m}$ satisfying the following equations

$$1) \mathbf{AXA} = \mathbf{A}; 2) \mathbf{XAX} = \mathbf{X}; 3) (\mathbf{AX})^* = \mathbf{AX}; 4) (\mathbf{XA})^* = \mathbf{XA}.$$

For $\mathbf{A} \in \mathbb{H}^{n \times n}$ with $k = \text{Ind } \mathbf{A}$ the smallest positive number such that $\text{rank } \mathbf{A}^{k+1} = \text{rank } \mathbf{A}^k$, the Drazin inverse of \mathbf{A} , denoted by \mathbf{A}^D , is defined to be the unique matrix \mathbf{X} that satisfying Eq. 2) and the equations

$$5) \mathbf{AX} = \mathbf{XA}; 6) \mathbf{A}^{k+1}\mathbf{X} = \mathbf{A}^k.$$

In particular, when $\text{Ind } \mathbf{A} = 1$, then \mathbf{X} is called the group inverse of \mathbf{A} and is denoted by $\mathbf{X} = \mathbf{A}^g$. If $\text{Ind } \mathbf{A} = 0$, then \mathbf{A} is nonsingular, and $\mathbf{A}^D \equiv \mathbf{A}^\dagger = \mathbf{A}^{-1}$.

Cline and Greville [17] extended the Drazin inverse of square matrix to rectangular matrix, that has been generalized to the quaternion algebra as follows.

For $\mathbf{A} \in \mathbb{H}^{m \times n}$ and $\mathbf{W} \in \mathbb{H}^{n \times m}$, the W-weighted Drazin inverse of \mathbf{A} with respect to \mathbf{W} is the unique solution to the following equations

$$7) (\mathbf{AW})^{k+1}\mathbf{XW} = (\mathbf{AW})^k; 8) \mathbf{XWAWX} = \mathbf{X}; 9) \mathbf{AWX} = \mathbf{XWA},$$

where $k = \max\{\text{Ind}(\mathbf{AW}), \text{Ind}(\mathbf{WA})\}$.

The Drazin inverse and weighted Drazin inverse has several important applications such as, applications in singular differential and difference equations [18], signal processing [19], Markov chains and statistic problems [20, 21], descriptor continuous-time systems [22], numerical analysis and Kronecker product systems [23], solving singular fuzzy linear system [24, 25], constrained linear systems [26] and etc.

Cramer's rule for the W-weighted Drazin inverse solutions, in particular, have been derived in [27] for singular linear equations and in [26] for a class of restricted matrix equations. Recently, within the framework of the theory of the column-row determinants Song [28] has first obtained a determinantal representation of the W-weighted Drazin inverse and Cramer's rule of a class of restricted matrix equations over the quaternion algebra. But in obtaining, he has used auxiliary matrices other than that are given. In [29], we have obtained new determinantal representations of the W-weighted Drazin inverse over the quaternion skew field without any auxiliary matrices.

An important application of determinantal representations of generalized inverses is the Cramer rule for generalized inverse solutions of matrix equations.

But, when there is a need for a W-weighted Drazin inverse solution? Consider for example the following matrix equation, $\mathbf{A}_1 \mathbf{X} = \mathbf{D}$. If \mathbf{A}_1 is rectangular and we can represent it as $\mathbf{A}_1 = \mathbf{WAW}$, where \mathbf{WA} and \mathbf{AW} are quadratic and singular, then its W-weighted Drazin inverse solution is needed.

In the paper we investigate analogs of Cramer's rule for W-weighted Drazin inverse solutions of the following matrix equations over the quaternion skew field \mathbb{H} ,

$$\mathbf{WAWX} = \mathbf{D}, \quad (1)$$

$$\mathbf{XWAW} = \mathbf{D}, \quad (2)$$

$$\mathbf{W}_1 \mathbf{AW}_1 \mathbf{XW}_2 \mathbf{BW}_2 = \mathbf{D}. \quad (3)$$

The paper is organized as follows. We start with some basic concepts and results from the theory of the row-column determinants in Subsection 2.1. In Subsection 2.2, we give determinantal representations of the Moore-Penrose and Drazin inverses for a quaternion matrix. Determinantal representations of the W-weighted Drazin inverse and its properties we consider in Subsection 2.3. In Subsection 3.1, we give the background of the problem of Cramer's rule for the W-weighted Drazin inverse solution. In Subsection 3.2 we obtain explicit representation formulas of the W-weighted Drazin inverse solutions (analogs of Cramer's rule) of the quaternion matrix equations (1-3). In Section 4, we give numerical examples to illustrate the main result.

2 Preliminaries

2.1 Elements of the theory of the column and row determinants

The theory of the row-column determinants over the quaternion skew field has been introduced in [30–32], and later it has been applied to research generalized inverses and generalized inverse solutions of matrix equations. In particular, determinantal representations of the Moore-Penrose [33, 34] and explicit representation formulas for the minimum norm least squares solutions of some quaternion matrix equations [35], and determinantal representations of the Drazin [36], and W-weighted Drazin inverses [29] have been obtained by the author. Song et al. derived determinantal representation of the generalized inverse $A_{T,S}^2$ [37], Bott-Duffin inverse [38] and the Cramer rule for the solutions of restricted matrix equations [39], and the generalized Stein quaternion matrix equation [40], etc.

For $\mathbf{A} = (a_{ij}) \in M(n, \mathbb{H})$ we define n row determinants and n column determinants as follows.

Suppose S_n is the symmetric group on the set $I_n = \{1, \dots, n\}$.

Definition 2.1 *The i -th row determinant of $\mathbf{A} = (a_{ij}) \in M(n, \mathbb{H})$ is defined for all $i = \overline{1, n}$ by putting*

$$\text{rdet}_i \mathbf{A} = \sum_{\sigma \in S_n} (-1)^{n-r} a_{i i_{k_1}} a_{i_{k_1} i_{k_1+1}} \dots a_{i_{k_1+l_1} i} \dots a_{i_{k_r} i_{k_r+1}} \dots a_{i_{k_r+l_r} i_{k_r}},$$

$$\sigma = (i i_{k_1} i_{k_1+1} \dots i_{k_1+l_1}) (i_{k_2} i_{k_2+1} \dots i_{k_2+l_2}) \dots (i_{k_r} i_{k_r+1} \dots i_{k_r+l_r}),$$

with conditions $i_{k_2} < i_{k_3} < \dots < i_{k_r}$ and $i_{k_t} < i_{k_t+s}$ for $t = \overline{2, r}$ and $s = \overline{1, l_t}$.

Definition 2.2 The j -th column determinant of $\mathbf{A} = (a_{ij}) \in M(n, \mathbb{H})$ is defined for all $j \in \overline{1, n}$ by putting

$$\text{cdet}_j \mathbf{A} = \sum_{\tau \in S_n} (-1)^{n-\tau} a_{j k_r, j k_r + l_r} \cdots a_{j k_r + 1, i k_r} \cdots a_{j j k_1 + l_1} \cdots a_{j k_1 + 1, j k_1} a_{j k_1 j},$$

$$\tau = (j k_r + l_r \cdots j k_r + 1 j k_r) \cdots (j k_2 + l_2 \cdots j k_2 + 1 j k_2) (j k_1 + l_1 \cdots j k_1 + 1 j k_1 j),$$

with conditions, $j k_2 < j k_3 < \cdots < j k_r$ and $j k_t < j k_t + s$ for $t = \overline{2, r}$ and $s = \overline{1, l_t}$.

Suppose \mathbf{A}^{ij} denotes the submatrix of \mathbf{A} obtained by deleting both the i th row and the j th column. Let $\mathbf{a}_{.j}$ be the j -th column and $\mathbf{a}_{i.}$ be the i -th row of \mathbf{A} . Suppose $\mathbf{A}_{.j}(\mathbf{b})$ denotes the matrix obtained from \mathbf{A} by replacing its j -th column with the column \mathbf{b} , and $\mathbf{A}_{i.}(\mathbf{b})$ denotes the matrix obtained from \mathbf{A} by replacing its i -th row with the row \mathbf{b} .

The following theorem has a key value in the theory of the column and row determinants.

Theorem 2.1 [30] If $\mathbf{A} = (a_{ij}) \in M(n, \mathbb{H})$ is Hermitian, then $\text{rdet}_1 \mathbf{A} = \cdots = \text{rdet}_n \mathbf{A} = \text{cdet}_1 \mathbf{A} = \cdots = \text{cdet}_n \mathbf{A} \in \mathbb{R}$.

Since all column and row determinants of a Hermitian matrix over \mathbb{H} are equal, we can define the determinant of a Hermitian matrix $\mathbf{A} \in M(n, \mathbb{H})$. By definition, we put, $\det \mathbf{A} := \text{rdet}_i \mathbf{A} = \text{cdet}_i \mathbf{A}$, for all $i \in \overline{1, n}$. The determinant of a Hermitian matrix has properties similar to a usual determinant. They are completely explored in [30, 31] by its row and column determinants. They can be summarized by the following theorems.

Theorem 2.2 If the i -th row of a Hermitian matrix $\mathbf{A} \in M(n, \mathbb{H})$ is replaced with a left linear combination of its other rows, i.e. $\mathbf{a}_{i.} = c_1 \mathbf{a}_{i_1.} + \cdots + c_k \mathbf{a}_{i_k.}$, where $c_l \in \mathbb{H}$ for all $l \in \overline{1, k}$ and $\{i, i_l\} \subset I_n$, then

$$\text{rdet}_i \mathbf{A}_{i.} (c_1 \mathbf{a}_{i_1.} + \cdots + c_k \mathbf{a}_{i_k.}) = \text{cdet}_i \mathbf{A}_{i.} (c_1 \mathbf{a}_{i_1.} + \cdots + c_k \mathbf{a}_{i_k.}) = 0.$$

Theorem 2.3 [30] If the j -th column of a Hermitian matrix $\mathbf{A} \in M(n, \mathbb{H})$ is replaced with a right linear combination of its other columns, i.e. $\mathbf{a}_{.j} = \mathbf{a}_{.j_1} c_1 + \cdots + \mathbf{a}_{.j_k} c_k$, where $c_l \in \mathbb{H}$ for all $l \in \overline{1, k}$ and $\{j, j_l\} \subset J_n$, then

$$\text{cdet}_j \mathbf{A}_{.j} (\mathbf{a}_{.j_1} c_1 + \cdots + \mathbf{a}_{.j_k} c_k) = \text{rdet}_j \mathbf{A}_{.j} (\mathbf{a}_{.j_1} c_1 + \cdots + \mathbf{a}_{.j_k} c_k) = 0.$$

The determinant of a Hermitian matrix also has a property of expansion along arbitrary rows and columns using row and column determinants of submatrices.

We have the following theorem on the determinantal representation of the inverse matrix over \mathbb{H} .

Theorem 2.4 [30] The necessary and sufficient condition of invertibility of $\mathbf{A} \in M(n, \mathbb{H})$ is $\text{ddet} \mathbf{A} \neq 0$. Then there exists $\mathbf{A}^{-1} = (\mathbf{L}\mathbf{A})^{-1} = (\mathbf{R}\mathbf{A})^{-1}$, where

$$(\mathbf{L}\mathbf{A})^{-1} = (\mathbf{A}^* \mathbf{A})^{-1} \mathbf{A}^* = \frac{1}{\text{ddet} \mathbf{A}} \begin{pmatrix} \mathbb{L}_{11} & \mathbb{L}_{21} & \cdots & \mathbb{L}_{n1} \\ \mathbb{L}_{12} & \mathbb{L}_{22} & \cdots & \mathbb{L}_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbb{L}_{1n} & \mathbb{L}_{2n} & \cdots & \mathbb{L}_{nn} \end{pmatrix}, \quad (4)$$

$$(R\mathbf{A})^{-1} = \mathbf{A}^* (\mathbf{A}\mathbf{A}^*)^{-1} = \frac{1}{\text{ddet}\mathbf{A}^*} \begin{pmatrix} \mathbb{R}_{11} & \mathbb{R}_{21} & \dots & \mathbb{R}_{n1} \\ \mathbb{R}_{12} & \mathbb{R}_{22} & \dots & \mathbb{R}_{n2} \\ \dots & \dots & \dots & \dots \\ \mathbb{R}_{1n} & \mathbb{R}_{2n} & \dots & \mathbb{R}_{nn} \end{pmatrix} \quad (5)$$

and

$$\mathbb{L}_{ij} = \text{cdet}_j(\mathbf{A}^*\mathbf{A})_{.j}(\mathbf{a}_{.i}^*), \quad \mathbb{R}_{ij} = \text{rdet}_i(\mathbf{A}\mathbf{A}^*)_{i.}(\mathbf{a}_{j.}^*),$$

for $i, j = \overline{1, n}$, and $\text{ddet}\mathbf{A} = \det(\mathbf{A}\mathbf{A}^*) = \det(\mathbf{A}^*\mathbf{A})$.

2.2 Determinantal representations of the Moore-Penrose and Drazin inverses over the quaternion skew field

We shall use the following notations. Let $\alpha := \{\alpha_1, \dots, \alpha_k\} \subseteq \{1, \dots, m\}$ and $\beta := \{\beta_1, \dots, \beta_k\} \subseteq \{1, \dots, n\}$ be subsets of the order $1 \leq k \leq \min\{m, n\}$. By \mathbf{A}_β^α denote the submatrix of \mathbf{A} determined by the rows indexed by α and the columns indexed by β . Then \mathbf{A}_α^α denotes the principal submatrix determined by the rows and columns indexed by α . If $\mathbf{A} \in M(n, \mathbb{H})$ is Hermitian, then by $|\mathbf{A}_\alpha^\alpha|$ denote the corresponding principal minor of $\det \mathbf{A}$. For $1 \leq k \leq n$, the collection of strictly increasing sequences of k integers chosen from $\{1, \dots, n\}$ is denoted by $L_{k,n} := \{\alpha : \alpha = (\alpha_1, \dots, \alpha_k), 1 \leq \alpha_1 \leq \dots \leq \alpha_k \leq n\}$. For fixed $i \in \alpha$ and $j \in \beta$, let $I_{r,m}\{i\} := \{\alpha : \alpha \in L_{r,m}, i \in \alpha\}$, $J_{r,n}\{j\} := \{\beta : \beta \in L_{r,n}, j \in \beta\}$.

Denote by $\mathbf{a}_{.j}^*$ and $\mathbf{a}_{i.}^*$ the j -th column and the i -th row of \mathbf{A}^* and by $\mathbf{a}_{.j}^{(m)}$ and $\mathbf{a}_{i.}^{(m)}$ the j -th column and the i -th row of \mathbf{A}^m , respectively.

The following theorem give determinantal representations of the Moore-Penrose inverse over the quaternion skew field \mathbb{H} .

Theorem 2.5 [32] *If $\mathbf{A} \in \mathbb{H}_r^{m \times n}$, then the Moore-Penrose inverse $\mathbf{A}^+ = (a_{ij}^+) \in \mathbb{H}^{n \times m}$ possess the following determinantal representations:*

$$a_{ij}^+ = \frac{\sum_{\beta \in J_{r,n}\{i\}} \text{cdet}_i((\mathbf{A}^*\mathbf{A})_{.i}(\mathbf{a}_{.j}^*))_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} |(\mathbf{A}^*\mathbf{A})_{\beta}^{\beta}|}, \quad (6)$$

or

$$a_{ij}^+ = \frac{\sum_{\alpha \in I_{r,m}\{j\}} \text{rdet}_j((\mathbf{A}\mathbf{A}^*)_{j.}(\mathbf{a}_{i.}^*))_{\alpha}^{\alpha}}{\sum_{\alpha \in I_{r,m}} |(\mathbf{A}\mathbf{A}^*)_{\alpha}^{\alpha}|}. \quad (7)$$

for all $i = \overline{1, n}$, $j = \overline{1, m}$.

Proposition 2.1 [20] *If $\text{Ind}(\mathbf{A}) = k$, then $\mathbf{A}^D = \mathbf{A}^k(\mathbf{A}^{2k+1})^+\mathbf{A}^k$.*

Denote by $\hat{\mathbf{a}}_{.s}$ and $\check{\mathbf{a}}_{t.}$ the s -th column of $(\mathbf{A}^{2k+1})^*\mathbf{A}^k =: \hat{\mathbf{A}} = (\hat{a}_{ij}) \in \mathbb{H}^{n \times n}$ and the t -th row of $\mathbf{A}^k(\mathbf{A}^{2k+1})^* =: \check{\mathbf{A}} = (\check{a}_{ij}) \in \mathbb{H}^{n \times n}$, respectively for all $s, t = \overline{1, n}$. Using the determinantal representations of the Moore-Penrose inverse (6) and (7), and Proposition 2.1 the following determinantal representations of the Drazin inverse for an arbitrary square matrix over \mathbb{H} have been obtained in [33].

Theorem 2.6 [33] If $\mathbf{A} \in M(n, \mathbb{H})$ with $\text{Ind } \mathbf{A} = k$ and $\text{rank } \mathbf{A}^{k+1} = \text{rank } \mathbf{A}^k = r$, then the Drazin inverse \mathbf{A}^D possess the determinantal representations,

$$a_{ij}^D = \frac{\sum_{t=1}^n a_{it}^{(k)} \sum_{\beta \in J_{r,n}\{t\}} \text{cdet}_t \left((\mathbf{A}^{2k+1})^* (\mathbf{A}^{2k+1})_{.t} (\hat{\mathbf{a}}_{.j}) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} \left| (\mathbf{A}^{2k+1})^* (\mathbf{A}^{2k+1})_{\beta}^{\beta} \right|} \quad (8)$$

and

$$a_{ij}^D = \frac{\sum_{s=1}^n \left(\sum_{\alpha \in I_{r,n}\{s\}} \text{rdet}_s \left((\mathbf{A}^{2k+1} (\mathbf{A}^{2k+1})^*)_{.s} (\check{\mathbf{a}}_{i.}) \right)_{\alpha}^{\alpha} \right) a_{sj}^{(k)}}{\sum_{\alpha \in I_{r,n}} \left| (\mathbf{A}^{2k+1} (\mathbf{A}^{2k+1})^*)_{\alpha}^{\alpha} \right|}. \quad (9)$$

In the special case, when $\mathbf{A} \in M(n, \mathbb{H})$ is Hermitian, we can obtain simpler determinantal representations of the Drazin inverse.

Theorem 2.7 [33] If $\mathbf{A} \in M(n, \mathbb{H})$ is Hermitian with $\text{Ind } \mathbf{A} = k$ and $\text{rank } \mathbf{A}^{k+1} = \text{rank } \mathbf{A}^k = r$, then the Drazin inverse $\mathbf{A}^D = (a_{ij}^D) \in \mathbb{H}^{n \times n}$ possess the following determinantal representations,

$$a_{ij}^D = \frac{\sum_{\beta \in J_{r,n}\{i\}} \text{cdet}_i \left((\mathbf{A}^{k+1})_{.i} (\mathbf{a}_{.j}^{(k)}) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} \left| (\mathbf{A}^{k+1})_{\beta}^{\beta} \right|}, \quad (10)$$

or

$$a_{ij}^D = \frac{\sum_{\alpha \in I_{r,n}\{j\}} \text{rdet}_j \left((\mathbf{A}^{k+1})_{j.} (\mathbf{a}_{i.}^{(k)}) \right)_{\alpha}^{\alpha}}{\sum_{\alpha \in I_{r,n}} \left| (\mathbf{A}^{k+1})_{\alpha}^{\alpha} \right|}. \quad (11)$$

2.3 Determinantal representations of the W-weighted Drazin inverse

Definition 2.3 For an arbitrary matrix over the quaternion skew field, $\mathbf{A} \in \mathbb{H}^{m \times n}$, we denote by

- $\mathcal{R}_r(\mathbf{A}) = \{\mathbf{y} \in \mathbb{H}^m : \mathbf{y} = \mathbf{A}\mathbf{x}, \mathbf{x} \in \mathbb{H}^n\}$, the column right space of \mathbf{A} ,
- $\mathcal{N}_r(\mathbf{A}) = \{\mathbf{y} \in \mathbb{H}^n : \mathbf{A}\mathbf{x} = 0\}$, the right null space of \mathbf{A} ,
- $\mathcal{R}_l(\mathbf{A}) = \{\mathbf{y} \in \mathbb{H}^n : \mathbf{y} = \mathbf{x}\mathbf{A}, \mathbf{x} \in \mathbb{H}^m\}$, the column left space of \mathbf{A} ,
- $\mathcal{N}_l(\mathbf{A}) = \{\mathbf{y} \in \mathbb{H}^m : \mathbf{x}\mathbf{A} = 0\}$, the left null space of \mathbf{A} .

We introduce some mathematical background from the theory of the W-weighted Drazin inverse [27, 41, 42] that can be generalized to \mathbb{H} .

Lemma 2.1 Let $\mathbf{A} \in \mathbb{H}^{m \times n}$ and $\mathbf{W} \in \mathbb{H}^{n \times m}$ with $k = \max\{\text{Ind}(\mathbf{AW}), \text{Ind}(\mathbf{WA})\}$.

Then we have:

- (a) $\mathbf{A}_{d,\mathbf{W}} = \mathbf{A} ((\mathbf{WA})^D)^2 = ((\mathbf{AW})^D)^2 \mathbf{A}$;
- (b) $\mathbf{A}_{d,\mathbf{W}} \mathbf{W} = (\mathbf{AW})^D$; $\mathbf{WA}_{d,\mathbf{W}} = (\mathbf{WA})^D$;
- (c) $\mathbf{A}_{d,\mathbf{W}} = \left\{ (\mathbf{AW})^k [(\mathbf{AW})^{2k+1}]^+ (\mathbf{AW})^k \right\} \mathbf{W}^+$;
 $\mathbf{A}_{d,\mathbf{W}} = \mathbf{W}^+ \left\{ (\mathbf{WA})^k [(\mathbf{WA})^{2k+1}]^+ (\mathbf{WA})^k \right\}$;
- (d) $\mathbf{WAWA}_{d,\mathbf{W}} = \mathbf{P}_{\mathcal{R}_r((\mathbf{WA})^k), \mathcal{N}_r((\mathbf{WA})^k)}$; $\mathbf{A}_{d,\mathbf{W}} \mathbf{WAW} = \mathbf{P}_{\mathcal{R}_l((\mathbf{AW})^k), \mathcal{N}_l((\mathbf{AW})^k)}$,

where $\mathbf{P}_{\mathcal{R}_r((\mathbf{WA})^k), \mathcal{N}_r((\mathbf{WA})^k)}$ is the projector on $\mathcal{R}_r((\mathbf{WA})^k)$ along $\mathcal{N}_r((\mathbf{WA})^k)$, and $\mathbf{P}_{\mathcal{R}_l((\mathbf{AW})^k), \mathcal{N}_l((\mathbf{AW})^k)}$ is the projector on $\mathcal{R}_l((\mathbf{AW})^k)$ along $\mathcal{N}_l((\mathbf{AW})^k)$.

In particular, the point (a) of Lemma 2.1 due to Cline and Greville [17] is generalized [28] to \mathbb{H} . Using this proposition, we have obtained [29] the following determinantal representations W-weighted Drazin inverse.

Denote $\mathbf{WA} =: \mathbf{U} = (u_{ij}) \in \mathbb{H}^{n \times n}$ and $\mathbf{AW} =: \mathbf{V} = (v_{ij}) \in \mathbb{H}^{m \times m}$.

Due to Theorem 2.6, we denote an entry of the Drazin inverse \mathbf{U}^D by

$$u_{ij}^{D,1} = \frac{\sum_{t=1}^n u_{it}^{(k)} \sum_{\beta \in J_{r,n}\{t\}} \text{cdet}_t \left((\mathbf{U}^{2k+1})^* (\mathbf{U}^{2k+1})_{.t} (\hat{\mathbf{u}}_{.j}) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r,n}} \left| (\mathbf{U}^{2k+1})^* (\mathbf{U}^{2k+1})_{\beta}^{\beta} \right|} \quad (12)$$

or

$$u_{ij}^{D,2} = \frac{\sum_{s=1}^n \left(\sum_{\alpha \in I_{r,n}\{s\}} \text{rdet}_s \left((\mathbf{U}^{2k+1} (\mathbf{U}^{2k+1})^*)_{.s} (\check{\mathbf{u}}_{i.}) \right)_{\alpha}^{\alpha} \right) u_{sj}^{(k)}}{\sum_{\alpha \in I_{r,n}} \left| (\mathbf{U}^{2k+1} (\mathbf{U}^{2k+1})^*)_{\alpha}^{\alpha} \right|} \quad (13)$$

where $\hat{\mathbf{u}}_{.s}$ and $\check{\mathbf{u}}_{.t}$ are the s -th column of $(\mathbf{U}^{2k+1})^* \mathbf{U}^k =: \hat{\mathbf{U}} = (\hat{u}_{ij}) \in \mathbb{H}^{n \times n}$ and the t -th row of $\mathbf{U}^k (\mathbf{U}^{2k+1})^* =: \check{\mathbf{U}} = (\check{u}_{ij}) \in \mathbb{H}^{n \times n}$, respectively for all $s, t = \overline{1, n}$, $r = \text{rank } \mathbf{U}^{k+1} = \text{rank } \mathbf{U}^k$.

Then we have the following determinantal representations of $\mathbf{A}_{d,\mathbf{W}} = (a_{ij}^{d,\mathbf{W}}) \in \mathbb{H}^{m \times n}$,

$$a_{ij}^{d,\mathbf{W}} = \sum_{q=1}^n a_{iq} (u_{qj}^D)^{(2)} \quad (14)$$

where

$$(u_{qj}^D)^{(2)} = \sum_{p=1}^n u_{qp}^{D,l} u_{pj}^{D,f} \quad (15)$$

for all $l, f = \overline{1, 2}$, and $u_{ij}^{D,1}$ from (12) and $u_{ij}^{D,2}$ from (13).

Similarly using $\mathbf{V} = (v_{ij}) \in \mathbb{H}^{m \times m}$,

$$a_{ij}^{d,\mathbf{W}} = \sum_{q=1}^m (v_{iq}^D)^{(2)} a_{qj}. \quad (16)$$

where the first factor is one of the four possible equations

$$(v_{iq}^D)^{(2)} = \sum_{p=1}^m v_{ip}^{D,l} v_{pq}^{D,f} \quad (17)$$

for all $l, f = \overline{1, 2}$, and an entry of the Drazin inverse \mathbf{V}^D is denoting by

$$v_{ij}^{D,1} = \frac{\sum_{t=1}^m v_{it}^{(k)} \sum_{\beta \in J_{r,m}\{t\}} \text{cdet}_t \left((\mathbf{V}^{2k+1})^* (\mathbf{V}^{2k+1})_{.t} (\hat{\mathbf{v}}_{.j}) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r,m}} \left| (\mathbf{V}^{2k+1})^* (\mathbf{V}^{2k+1})_{\beta}^{\beta} \right|} \quad (18)$$

or

$$v_{ij}^{D,2} = \frac{\sum_{s=1}^m \left(\sum_{\alpha \in I_{r,m}\{s\}} \text{rdet}_s \left((\mathbf{V}^{2k+1} (\mathbf{V}^{2k+1})^*)_{.s} (\check{\mathbf{v}}_{i.}) \right)_{\alpha}^{\alpha} \right) v_{sj}^{(k)}}{\sum_{\alpha \in I_{r,m}} \left| (\mathbf{V}^{2k+1} (\mathbf{V}^{2k+1})^*)_{\alpha}^{\alpha} \right|}, \quad (19)$$

where $\hat{\mathbf{v}}_{.s}$ and $\check{\mathbf{v}}_{t.}$ are the s -th column of $(\mathbf{V}^{2k+1})^* \mathbf{V}^k =: \hat{\mathbf{V}} = (\hat{v}_{ij}) \in \mathbb{H}^{m \times m}$ and the t -th row of $\mathbf{V}^k (\mathbf{V}^{2k+1})^* =: \check{\mathbf{V}} = (\check{v}_{ij}) \in \mathbb{H}^{m \times m}$, respectively for all $s, t = \overline{1, m}$, $r = \text{rank } \mathbf{V}^{k+1} = \text{rank } \mathbf{V}^k$.

The point (c) of Lemma 2.1 due to [23] has been generalized to \mathbb{H} in [33]. Using this proposition, we have obtained the following two determinantal representations of the \mathbf{W} -weighted Drazin inverse.

Theorem 2.8 [29] *Let $\mathbf{A} \in \mathbb{H}^{m \times n}$ and $\mathbf{W} \in \mathbb{H}_{r_1}^{n \times m}$ with $k = \text{Ind}(\mathbf{A}\mathbf{W})$ and $r = \text{rank}(\mathbf{A}\mathbf{W})^{k+1} = \text{rank}(\mathbf{A}\mathbf{W})^k$. Then the \mathbf{W} -weighted Drazin inverse of \mathbf{A} with respect to \mathbf{W} possesses the determinantal representations*

$$a_{ij}^{d,\mathbf{W}} = \frac{\sum_{t=1}^m \sum_{\alpha \in I_{r,m}\{t\}} \text{rdet}_t \left((\mathbf{V}^{2k+1} (\mathbf{V}^{2k+1})^*)_{.t} (\check{\mathbf{v}}_{i.}) \right)_{\alpha}^{\alpha} \sum_{\alpha \in I_{r_1,n}\{j\}} \text{rdet}_j \left((\mathbf{W}\mathbf{W}^*)_{.j} (\check{\mathbf{w}}_{t.}) \right)_{\alpha}^{\alpha}}{\sum_{\alpha \in I_{r,m}} \left| (\mathbf{V}^{2k+1} (\mathbf{V}^{2k+1})^*)_{\alpha}^{\alpha} \right| \sum_{\alpha \in I_{r_1,n}} \left| (\mathbf{W}\mathbf{W}^*)_{\alpha}^{\alpha} \right|} \quad (20)$$

and

$$a_{ij}^{d,\mathbf{W}} = \frac{\sum_{t=1}^n \sum_{\beta \in J_{r_1,m}\{i\}} \text{cdet}_i \left((\mathbf{W}^* \mathbf{W})_{.t} (\hat{\mathbf{w}}_{.t}) \right)_{\beta}^{\beta} \sum_{\beta \in J_{r,n}\{t\}} \text{cdet}_t \left(((\mathbf{U}^{2k+1})^* \mathbf{U}^{2k+1})_{.t} (\hat{\mathbf{u}}_{.j}) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r_1,m}} \left| (\mathbf{W}^* \mathbf{W})_{\beta}^{\beta} \right| \sum_{\beta \in J_{r,n}} \left| ((\mathbf{U}^{2k+1})^* \mathbf{U}^{2k+1})_{\beta}^{\beta} \right|} \quad (21)$$

where $\check{\mathbf{V}} = \mathbf{V}^k (\mathbf{V}^{2k+1})^*$, $\check{\mathbf{W}} = \mathbf{V}^k \mathbf{W}^*$, and $\hat{\mathbf{U}} = (\mathbf{U}^{2k+1})^* \mathbf{U}^k$, $\hat{\mathbf{W}} = \mathbf{W}^* \mathbf{U}^k$.

In the special cases, when $\mathbf{A}\mathbf{W} \in \mathbb{H}^{m \times m}$ and $\mathbf{W}\mathbf{A} \in \mathbb{H}^{n \times n}$ are Hermitian, we can obtain simpler determinantal representations of the \mathbf{W} -weighted Drazin inverse.

Theorem 2.9 [29] If $\mathbf{A} \in \mathbb{H}^{m \times n}$, $\mathbf{W} \in \mathbb{H}^{n \times m}$, and $\mathbf{AW} \in \mathbb{H}^{m \times m}$ is Hermitian with $k = \max\{\text{Ind}(\mathbf{AW}), \text{Ind}(\mathbf{WA})\}$ and $\text{rank}(\mathbf{AW})^{k+1} = \text{rank}(\mathbf{AW})^k = r$, then the W -weighted Drazin inverse $\mathbf{A}_{d,W} = (a_{ij}^{d,W}) \in \mathbb{H}^{m \times n}$ with respect to \mathbf{W} possess the following determinantal representations:

$$a_{ij}^{d,W} = \frac{\sum_{\beta \in J_{r,m}\{i\}} \text{cdet}_i \left((\mathbf{AW})_{\cdot i}^{k+2} (\bar{\mathbf{v}}_{\cdot j}) \right) \beta}{\sum_{\beta \in J_{r,m}} \left| (\mathbf{AW})_{\cdot i}^{k+2} \beta \right|}, \quad (22)$$

where $\bar{\mathbf{v}}_{\cdot j}$ is the j th column of $\bar{\mathbf{V}} = (\mathbf{AW})^k \mathbf{A}$ for all $j = \overline{1, m}$.

Theorem 2.10 [29] If $\mathbf{A} \in \mathbb{H}^{m \times n}$, $\mathbf{W} \in \mathbb{H}^{n \times m}$, and $\mathbf{WA} \in \mathbb{H}^{n \times n}$ is Hermitian with $k = \max\{\text{Ind}(\mathbf{AW}), \text{Ind}(\mathbf{WA})\}$ and $\text{rank}(\mathbf{WA})^{k+1} = \text{rank}(\mathbf{WA})^k = r$, then the W -weighted Drazin inverse $\mathbf{A}_{d,W} = (a_{ij}^{d,W}) \in \mathbb{H}^{m \times n}$ with respect to \mathbf{W} possess the following determinantal representations:

$$a_{ij}^{d,W} = \frac{\sum_{\alpha \in I_{r,n}\{j\}} \text{rdet}_j \left((\mathbf{WA})_{j \cdot}^{k+2} (\bar{\mathbf{u}}_{i \cdot}^{(k)}) \right) \alpha}{\sum_{\alpha \in I_{r,n}} \left| (\mathbf{WA})_{j \cdot}^{k+2} \alpha \right|}. \quad (23)$$

where $\bar{\mathbf{u}}_{i \cdot}$ is the i th row of $\bar{\mathbf{U}} = \mathbf{A}(\mathbf{WA})^k$ for all $i = \overline{1, n}$.

3 Cramer's rule for the W -weighted Drazin inverse solution

3.1 Background of the problem

In [27] Wei has established a Cramer's rule for solving of a general restricted equation

$$\mathbf{WAWx} = \mathbf{b}, \quad \mathbf{x} \in \mathcal{R}[(\mathbf{AW})^{k_1}], \quad (24)$$

where $\mathbf{A} \in \mathbb{C}^{m \times n}$, $\mathbf{W} \in \mathbb{C}^{n \times m}$ with $\text{Ind}(\mathbf{AW}) = k_1$, $\text{Ind}(\mathbf{WA}) = k_2$ and $\text{rank}(\mathbf{AW})^{k_1} = r_1$, $\text{rank}(\mathbf{WA})^{k_2} = r_2$. He proofed that the restricted matrix equation (24) has a unique solution, $\mathbf{x} = \mathbf{A}_{d,W} \mathbf{b}$, and presented its Cramer's rule as follows,

$$x_j = \det \begin{pmatrix} \mathbf{WAW}(j \rightarrow \mathbf{b}) & \mathbf{U}_1 \\ \mathbf{V}_1(j \rightarrow 0) & 0 \end{pmatrix} / \det \begin{pmatrix} \mathbf{WAW} & \mathbf{U}_1 \\ \mathbf{V}_1 & 0 \end{pmatrix}, \quad (25)$$

where $\mathbf{U}_1 \in \mathbb{C}^{n \times n-r_2}$, $\mathbf{V}_1^* \in \mathbb{C}^{m \times m-r_1}$ are matrices whose columns form bases for $\mathcal{N}((\mathbf{WA})^{k_2})$ and $\mathcal{N}((\mathbf{AW})^{k_1})$, respectively.

Recently, within the framework of a theory of the column and row determinants Song [28] has considered the characterization of the W -weighted Drazin inverse over the quaternion skew and presented a Cramer's rule of the restricted matrix equation,

$$\mathbf{W}_1 \mathbf{AW}_1 \mathbf{XW}_2 \mathbf{BW}_2 = \mathbf{D}, \quad (26)$$

$$\mathcal{R}_r(\mathbf{X}) \subset \mathcal{R}_r((\mathbf{AW}_1)^{k_1}), \quad \mathcal{N}_r(\mathbf{X}) \supset \mathcal{N}_r((\mathbf{W}_2 \mathbf{B})^{k_2}), \quad (27)$$

$$\mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l((\mathbf{BW}_2)^{k_2}), \quad \mathcal{N}_l(\mathbf{X}) \supset \mathcal{N}_l((\mathbf{W}_1 \mathbf{A})^{k_1}),$$

where $\mathbf{A} \in \mathbb{H}^{m \times n}$, $\mathbf{W}_1 \in \mathbb{H}^{n \times m}$, $\mathbf{B} \in \mathbb{H}^{p \times q}$, $\mathbf{W}_2 \in \mathbb{H}^{q \times p}$, and $\mathbf{D} \in \mathbb{H}^{n \times p}$ with $k_1 = \max \{ \text{Ind}(\mathbf{A}\mathbf{W}_1), \text{Ind}(\mathbf{W}_1\mathbf{A}) \}$, $k_2 = \max \{ \text{Ind}(\mathbf{B}\mathbf{W}_2), \text{Ind}(\mathbf{W}_2\mathbf{B}) \}$, and $\text{rank}(\mathbf{A}\mathbf{W}_1)^{k_1} = s_1$, $\text{rank}(\mathbf{B}\mathbf{W}_2)^{k_2} = s_2$.

He proofed that if

$$\mathbf{D} \in \mathcal{R}_r((\mathbf{W}_1\mathbf{A})^{k_1}, (\mathbf{W}_2\mathbf{B})^{k_2}), \mathbf{D} \in \mathcal{R}_l((\mathbf{A}\mathbf{W}_1)^{k_1}, (\mathbf{B}\mathbf{W}_2)^{k_2})$$

and there exist auxiliary matrices of full column rank, $\mathbf{L}_1 \in \mathbb{H}_{n-s_1}^{n \times n-s_1}$, $\mathbf{M}_1^* \in \mathbb{H}_{m-s_1}^{m \times m-s_1}$, $\mathbf{L}_2 \in \mathbb{H}_{q-s_2}^{q \times q-s_2}$, $\mathbf{M}_2^* \in \mathbb{H}_{p-s_2}^{p \times p-s_2}$ with additional terms of their ranges and null spaces, then the restricted matrix equation (26) has a unique solution,

$$\mathbf{X} = \mathbf{A}_{d, \mathbf{W}_1} \mathbf{D} \mathbf{B}_{d, \mathbf{W}_2}.$$

Using auxiliary matrices, \mathbf{L}_1 , \mathbf{M}_1 , \mathbf{L}_2 , \mathbf{M}_2 , Song presented its Cramer's rule by analogy to (25).

In this paper we have avoided such approach and have obtained explicit formulas for determinantal representations of the W-weighted Drazin inverse solutions of matrix equations by using only given matrices.

3.2 A Cramer's rule for the W-weighted Drazin inverse solutions of some matrix equations

Consider the following restricted matrix equation,

$$\mathbf{W}\mathbf{A}\mathbf{W}\mathbf{X} = \mathbf{D}, \quad (28)$$

$$\mathcal{R}_r(\mathbf{X}) \subset \mathcal{R}_r((\mathbf{A}\mathbf{W})^k), \mathcal{N}_l(\mathbf{X}) \supset \mathcal{N}_l((\mathbf{W}\mathbf{A})^k), \quad (29)$$

where $\mathbf{A} \in \mathbb{H}^{m \times n}$, $\mathbf{W} \in \mathbb{H}_{r_1}^{n \times m}$ with $k = \max \{ \text{Ind}(\mathbf{A}\mathbf{W}), \text{Ind}(\mathbf{W}\mathbf{A}) \}$, and $\mathbf{D} \in \mathbb{H}^{n \times p}$.

Theorem 3.1 *If $\mathbf{D} \in \mathcal{R}_r((\mathbf{A}\mathbf{W})^k)$ and $\mathbf{D} \supset \mathcal{N}_l((\mathbf{W}\mathbf{A})^k)$, then the restricted matrix equation (28) has a unique solution,*

$$\mathbf{X} = \mathbf{A}_{d, \mathbf{W}} \mathbf{D}, \quad (30)$$

which possess the following determinantal representations for all $i = \overline{1, m}$, $j = \overline{1, p}$,
i)

$$x_{ij} = \frac{\sum_{t=1}^n \sum_{\beta \in J_{r_1, m} \{i\}} \text{cdet}_i((\mathbf{W}^* \mathbf{W})_{.t} (\hat{\mathbf{w}}_{.t}))_{\beta}^{\beta} \sum_{\beta \in J_{r, n} \{t\}} \text{cdet}_t \left(\left((\mathbf{U}^{2k+1})^* \mathbf{U}^{2k+1} \right)_{.t} (\hat{\mathbf{d}}_{.j}) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r_1, m}} \left| (\mathbf{W}^* \mathbf{W})_{\beta}^{\beta} \right| \sum_{\beta \in J_{r, n}} \left| ((\mathbf{U}^{2k+1})^* \mathbf{U}^{2k+1})_{\beta}^{\beta} \right|} \quad (31)$$

where $\hat{\mathbf{d}}_{.j}$ is the j -th column of $\hat{\mathbf{D}} = \hat{\mathbf{U}} \mathbf{D} = (\mathbf{U}^{2k+1})^ \mathbf{U}^k \mathbf{D}$, $\mathbf{U} = \mathbf{W}\mathbf{A}$, $\hat{\mathbf{W}} = \mathbf{W}^* \mathbf{U}^k$, and $r = \text{rank}(\mathbf{W}\mathbf{A})^{k+1} = \text{rank}(\mathbf{W}\mathbf{A})^k$.*
ii)

$$x_{ij} = \sum_{q=1}^m (v_{iq}^D)^{(2)} r_{qj}, \quad (32)$$

where $(v_{iq}^D)^{(2)}$ can be obtained by (17) and $\mathbf{AD} = \mathbf{R} = (r_{qj}) \in \mathbb{H}^{m \times p}$.
iii) If $\mathbf{AW} \in \mathbb{H}^{m \times m}$ is Hermitian, then

$$x_{ij} = \frac{\sum_{\beta \in J_{r,m}\{i\}} \text{cdet}_i \left((\mathbf{AW})_{.i}^{k+2} (\mathbf{f}_j) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r,m}} \left| (\mathbf{AW})^{k+2} \right|_{\beta}^{\beta}}, \quad (33)$$

where \mathbf{f}_j is the j -th column of $\mathbf{F} = \bar{\mathbf{V}}\mathbf{D} = (\mathbf{AW})^k \mathbf{AD}$.

Proof. The proof contains two parts. We first shall establish that the unique solution of (28) can be represented as (30). By the definition of the right range, we have

$$\mathbf{D} = (\mathbf{WA})^k \mathbf{Y}$$

for some matrix $\mathbf{Y} \in \mathbb{H}^{n \times p}$. It follows that

$$\mathcal{R}_r(\mathbf{D}) \subset \mathcal{R}_r((\mathbf{WA})^k).$$

Then by Lemma 2.1 (d),

$$\mathbf{WAWA}_{d,W} \mathbf{D} = \mathbf{P}_{\mathcal{R}_r((\mathbf{WA})^k), \mathcal{N}_r((\mathbf{WA})^k)} \mathbf{D} = \mathbf{D}.$$

It means that (30) is a solution of (28) and satisfies the restricted conditions (29).

Now we prove the uniqueness of (30). Let \mathbf{X}_0 is a solution of (28). Then it satisfies the restricted conditions (29), and

$$\mathbf{A}_{d,W} \mathbf{D} = \mathbf{A}_{d,W} \mathbf{WAWX}_0 = \mathbf{P}_{\mathcal{R}_r((\mathbf{WA})^k), \mathcal{N}_r((\mathbf{WA})^k)} \mathbf{X}_0 = \mathbf{X}_0.$$

To derive a Cramer's rule (31), we use the determinantal representation (21) for $\mathbf{A}_{d,W}$. Then

$$x_{ij} = \sum_{s=1}^p a_{is}^{d,W} d_{sj} = \sum_{s=1}^p \left[\frac{\sum_{t=1}^n \sum_{\beta \in J_{r_1,m}\{i\}} \text{cdet}_i(\mathbf{W}^* \mathbf{W})_{.t} (\hat{\mathbf{w}}_{.t})_{\beta}^{\beta} \sum_{\beta \in J_{r,n}\{t\}} \text{cdet}_t \left((\mathbf{U}^{2k+1})^* \mathbf{U}^{2k+1} \right)_{.t} (\hat{\mathbf{u}}_{.s})_{\beta}^{\beta}}{\sum_{\beta \in J_{r_1,m}} \left| (\mathbf{W}^* \mathbf{W}) \right|_{\beta}^{\beta} \sum_{\beta \in J_{r,n}} \left| (\mathbf{U}^{2k+1})^* \mathbf{U}^{2k+1} \right|_{\beta}^{\beta}} \right] d_{sj} \quad (34)$$

Denote $\hat{\mathbf{D}} = \hat{\mathbf{U}}\mathbf{D} = (\mathbf{U}^{2k+1})^* \mathbf{U}^k \mathbf{D}$, where $\hat{\mathbf{D}} = (\hat{d}_{sj}) \in \mathbb{H}^{n \times p}$. Since

$$\sum_{s=1}^p \hat{\mathbf{u}}_{.s} d_{sj} = \hat{\mathbf{d}}_{.j},$$

where $\hat{\mathbf{d}}_{.j}$ is the j -th column of $\hat{\mathbf{D}}$, then (31) follows from (34).

Similarly, we derive the analogs of Cramer's rule (32) and (33) by using the determinantal representations for the \mathbf{W} -weighted Drazin inverse (16), and (22), respectively. \square

Remark 3.1 In the complex case, i.e. $\mathbf{A} \in \mathbb{C}^{m \times n}$, $\mathbf{W} \in \mathbb{C}_{r_1}^{n \times m}$, and $\mathbf{D} \in \mathbb{C}^{n \times p}$, we substitute usual determinants for all corresponding row and column determinants in (31), (32), and (33).

Note that in the case iii), the condition $\mathbf{AW} \in \mathbb{C}^{m \times m}$ be Hermitian is not necessary, then in the complex case (33) will have the form

$$x_{ij} = \frac{\sum_{\beta \in J_{r,m}\{i\}} \left| \left((\mathbf{AW})_{\cdot i}^{k+2} (\mathbf{f}_j) \right)_{\beta} \right|}{\sum_{\beta \in J_{r,m}} \left| (\mathbf{AW})_{\cdot i}^{k+2} \right|_{\beta}},$$

where \mathbf{f}_j is the j -th column of $\mathbf{F} = \bar{\mathbf{V}}\mathbf{D} = (\mathbf{AW})^k \mathbf{AD}$.

Now, consider the following restricted matrix equation,

$$\mathbf{XWAW} = \mathbf{D}, \quad (35)$$

$$\mathcal{R}_l(\mathbf{X}) \subset \mathcal{R}_l((\mathbf{AW})^k), \quad \mathcal{N}_r(\mathbf{X}) \supset \mathcal{N}_r((\mathbf{WA})^k), \quad (36)$$

where $\mathbf{A} \in \mathbb{H}^{m \times n}$, $\mathbf{W} \in \mathbb{H}_{r_1}^{n \times m}$ with $k = \max\{\text{Ind}(\mathbf{AW}), \text{Ind}(\mathbf{WA})\}$, and $\mathbf{D} \in \mathbb{H}^{q \times m}$.

Theorem 3.2 If $\mathbf{D} \in \mathcal{R}_l((\mathbf{AW})^k)$ and $\mathbf{D} \in \mathcal{N}_r((\mathbf{WA})^k)$, then the restricted matrix equation (35) has a unique solution,

$$\mathbf{X} = \mathbf{DA}_{d,W}, \quad (37)$$

which possess the following determinantal representations for $i = \overline{1, q}$, $j = \overline{1, n}$,
i)

$$x_{ij} = \frac{\sum_{l=1}^m \sum_{\alpha \in I_{r,m}\{t\}} \text{rdet}_l \left(\left(\mathbf{V}^{2k+1} (\mathbf{V}^{2k+1})^* \right)_l (\check{\mathbf{d}}_{\cdot}) \right)_{\alpha} \sum_{\alpha \in I_{r_1,n}\{j\}} \text{rdet}_j \left((\mathbf{WW}^*)_{\cdot j} (\check{\mathbf{w}}_l) \right)_{\alpha}}{\sum_{\alpha \in I_{r,m}} \left| \left(\mathbf{V}^{2k+1} (\mathbf{V}^{2k+1})^* \right)_{\cdot} \right|_{\alpha} \sum_{\alpha \in I_{r_1,n}} \left| (\mathbf{WW}^*)_{\cdot} \right|_{\alpha}} \quad (38)$$

where $\check{\mathbf{d}}_{\cdot}$ is the i -th row of $\check{\mathbf{D}} = \mathbf{D}\check{\mathbf{V}} = \mathbf{D}\mathbf{V}^k(\mathbf{V}^{2k+1})^*$, $\mathbf{V} = \mathbf{AW}$ and $r = \text{rank}(\mathbf{AW})^{k+1} = \text{rank}(\mathbf{AW})^k$.

ii)

$$x_{ij} = \sum_{t=1}^n l_{it} (u_{tj}^D)^{(2)}, \quad (39)$$

where $(u_{qj}^D)^{(2)}$ can be obtained by (15) and $\mathbf{DA} = \mathbf{L} = (l_{it}) \in \mathbb{H}^{q \times n}$.

iii) If $\mathbf{AW} \in \mathbb{H}^{m \times m}$ is Hermitian, then

$$x_{ij} = \frac{\sum_{\alpha \in I_{r,n}\{j\}} \text{rdet}_j \left((\mathbf{WA})_{\cdot j}^{k+2} (\mathbf{g}_i) \right)_{\alpha}}{\sum_{\alpha \in I_{r,n}} \left| (\mathbf{WA})_{\cdot j}^{k+2} \right|_{\alpha}}. \quad (40)$$

where \mathbf{g}_i is the i -th row of $\mathbf{G} = \mathbf{DA}(\mathbf{WA})^k$ for all $i = \overline{1, n}$.

Proof. The proof is similar to the proof of Theorem 3.1. \square

Remark 3.2 In the complex case, i.e. $\mathbf{A} \in \mathbb{C}^{m \times n}$, $\mathbf{W} \in \mathbb{C}_{r_1}^{n \times m}$, and $\mathbf{D} \in \mathbb{C}^{n \times p}$, we substitute usual determinants for all corresponding row and column determinants in (38), (39), and (40). Herein the condition $\mathbf{WA} \in \mathbb{C}^{n \times n}$ be Hermitian is not necessary, then in the complex case (40) can be represented as follows,

$$x_{ij} = \frac{\sum_{\alpha \in I_{r,n}\{j\}} |((\mathbf{WA})_{j\cdot}^{k+2}(\mathbf{g}_{i\cdot}))_{\alpha}|}{\sum_{\alpha \in I_{r,n}} |(\mathbf{WA})_{\cdot}^{k+2}{}_{\alpha}|}.$$

where $\mathbf{g}_{i\cdot}$ is the i -th row of $\mathbf{G} = \mathbf{DA}(\mathbf{WA})^k$ for all $i = \overline{1, n}$.

Now we consider the matrix equation (26) with the constraints (3.1). Denote $\mathbf{ADB} =: \tilde{\mathbf{D}} = (\tilde{d}_{lf}) \in \mathbb{H}^{m \times q}$, and $\bar{\mathbf{V}}\mathbf{D}\bar{\mathbf{U}} =: \bar{\mathbf{D}} = (\bar{d}_{lf}) \in \mathbb{H}^{m \times q}$, where $\bar{\mathbf{V}} := (\mathbf{AW}_1)^{k_1} \mathbf{A}$, $\bar{\mathbf{U}} := \mathbf{B}(\mathbf{W}_2\mathbf{B})^{k_2}$.

Theorem 3.3 Suppose $\mathbf{D} \in \mathbb{H}^{n \times p}$, $\mathbf{A} \in \mathbb{H}^{m \times n}$, $\mathbf{W}_1 \in \mathbb{H}_{r_1}^{n \times m}$ with $k_1 = \max\{\text{Ind}(\mathbf{AW}_1), \text{Ind}(\mathbf{W}_1\mathbf{A})\}$, where $\text{rank}(\mathbf{AW}_1)^{k_1} = s_1$, and $\mathbf{B} \in \mathbb{H}^{p \times q}$, $\mathbf{W}_2 \in \mathbb{H}_{r_2}^{q \times p}$ with $k_2 = \max\{\text{Ind}(\mathbf{BW}_2), \text{Ind}(\mathbf{W}_2\mathbf{B})\}$, $\text{rank}(\mathbf{BW}_2)^{k_2} = s_2$. If $\mathbf{D} \in \mathcal{R}_r((\mathbf{W}_1\mathbf{A})^{k_1}, (\mathbf{W}_2\mathbf{B})^{k_2})$, $\mathbf{D} \in \mathcal{R}_l((\mathbf{AW}_1)^{k_1}, (\mathbf{BW}_2)^{k_2})$, then the restricted matrix equation (26) has a unique solution,

$$\mathbf{X} = \mathbf{A}_{d, \mathbf{W}_1} \mathbf{D} \mathbf{B}_{d, \mathbf{W}_2}, \quad (41)$$

which possess the following determinantal representations for all $i = \overline{1, m}$, $j = \overline{1, q}$.

i)

$$x_{ij} = \sum_{l=1}^m \sum_{f=1}^q (v_{il}^D)^{(2)} \tilde{d}_{lf} (u_{fj}^D)^{(2)}, \quad (42)$$

where $(v_{il}^D) = \mathbf{V}^D$ is the Drazin inverse of $\mathbf{V} = \mathbf{AW}_1$ and $(v_{il}^D)^{(2)}$ can be obtained by (17), and $(u_{fj}^D) = \mathbf{U}^D$ is the Drazin inverse of $\mathbf{U} = \mathbf{W}_2\mathbf{B}$ and $(u_{fj}^D)^{(2)}$ can be obtained by (15).

ii) If $\mathbf{AW}_1 \in \mathbb{H}^{m \times m}$ and $\mathbf{W}_2\mathbf{B} \in \mathbb{H}^{q \times q}$ are Hermitian, then

$$x_{ij} = \frac{\sum_{\beta \in J_{s_1, m}\{i\}} \text{cdet}_i \left((\mathbf{AW}_1)_{\cdot i}^{k_1+2} (\mathbf{d}_{\cdot j}^{\mathbf{B}})_{\beta}^{\beta} \right)}{\sum_{\beta \in J_{s_1, m}} |(\mathbf{AW}_1)^{k_1+2}{}_{\beta}| \sum_{\alpha \in I_{s_2, q}} |(\mathbf{W}_2\mathbf{B})^{k_2+2}{}_{\alpha}|}, \quad (43)$$

or

$$x_{ij} = \frac{\sum_{\alpha \in I_{s_2, q}\{j\}} \text{rdet}_j \left((\mathbf{W}_2\mathbf{B})_{j\cdot}^{k_2+2} (\mathbf{d}_i^{\mathbf{A}})_{\alpha}^{\alpha} \right)}{\sum_{\beta \in J_{s_1, m}} |(\mathbf{AW}_1)^{k_1+2}{}_{\beta}| \sum_{\alpha \in I_{s_2, q}} |(\mathbf{W}_2\mathbf{B})^{k_2+2}{}_{\alpha}|}, \quad (44)$$

where

$$\mathbf{d}_{\cdot j}^{\mathbf{B}} = \left(\sum_{\alpha \in I_{s_2, q}\{j\}} \text{rdet}_j \left((\mathbf{W}_2\mathbf{B})_{j\cdot}^{k_2+2} (\bar{\mathbf{d}}_{t\cdot})_{\alpha}^{\alpha} \right) \right) \in \mathbb{H}^{n \times 1}, \quad t = \overline{1, n} \quad (45)$$

$$\mathbf{d}_{i\cdot}^{\mathbf{A}} = \left(\sum_{\beta \in J_{s_1, m}\{i\}} \text{cdet}_i \left((\mathbf{A}\mathbf{W}_1)^{k_1+2} (\bar{\mathbf{d}}_{\cdot l}) \right)_{\beta}^{\beta} \right) \in \mathbb{H}^{1 \times q}, \quad l = \overline{1, q} \quad (46)$$

are the column vector and the row vector, respectively. $\bar{\mathbf{d}}_{i\cdot}$ and $\bar{\mathbf{d}}_{\cdot j}$ are the i -th row and the j -th column of $\bar{\mathbf{D}}$ for all $i = \overline{1, n}$, $j = \overline{1, p}$.

Proof. The existence and uniqueness of the solution (41) can be proved similar as in ([28], Theorem 5.2).

To establish a Cramer's rule of (26) we note that we shall not use the determinantal representations (22) and (22) for (41) because corresponding determinantal representations of it's solution will be too cumbersome.

To derive a Cramer's rule (42) we use the sentence (a) from Lemma 2.1. Then we obtain

$$\mathbf{X} = ((\mathbf{A}\mathbf{W}_1)^D)^2 \mathbf{A} \mathbf{D} \mathbf{B} ((\mathbf{W}_2 \mathbf{B})^D)^2. \quad (47)$$

Denote $\mathbf{A} \mathbf{D} \mathbf{B} =: \tilde{\mathbf{D}} = (\tilde{d}_{lf}) \in \mathbb{H}^{m \times q}$, $\mathbf{V} := \mathbf{A}\mathbf{W}_1$, and $\mathbf{U} := \mathbf{W}_2 \mathbf{B}$. Then the equation (47) will be written component-wise as follows

$$x_{ij} = \sum_{s=1}^p \sum_{t=1}^n (a_{it}^{d, W_1}) d_{ts} (b_{sj}^{d, W_2}) = \sum_{s=1}^p \sum_{t=1}^n \left(\sum_{l=1}^m (v_{il}^D)^{(2)} a_{lt} \right) d_{ts} \left(\sum_{f=1}^q b_{sf} (u_{fj}^D)^{(2)} \right)$$

By changing the order of summation, from here it follows (42).

ii) If $\mathbf{A} \in \mathbb{H}_{r_1}^{m \times n}$, $\mathbf{B} \in \mathbb{H}_{r_2}^{p \times q}$ and $\mathbf{A}\mathbf{W}_1 \in \mathbb{H}^{m \times m}$ and $\mathbf{W}_2 \mathbf{B} \in \mathbb{H}^{q \times q}$ are Hermitian, then by Theorems 2.9 and 2.10 the \mathbf{W} -weighted Drazin inverses $\mathbf{A}_{d, W_1} = (a_{ij}^{d, W_1}) \in \mathbb{H}^{m \times n}$ and $\mathbf{B}_{d, W_2} = (b_{ij}^{d, W_2}) \in \mathbb{H}^{q \times p}$ posses the following determinantal representations respectively,

$$a_{ij}^{d, W_1} = \frac{\sum_{\beta \in J_{s_1, m}\{i\}} \text{cdet}_i \left((\mathbf{A}\mathbf{W}_1)^{k_1+2} (\bar{\mathbf{v}}_{\cdot j}) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r, m}} \left| (\mathbf{A}\mathbf{W}_1)^{k_1+2} \right|_{\beta}^{\beta}}, \quad (48)$$

where $\bar{\mathbf{v}}_{\cdot j}$ is the j -th column of $\bar{\mathbf{V}} = (\mathbf{A}\mathbf{W}_1)^{k_1} \mathbf{A}$ for all $j = \overline{1, m}$;

$$b_{ij}^{d, W_2} = \frac{\sum_{\alpha \in I_{s_2, q}\{j\}} \text{rdet}_j \left((\mathbf{W}_2 \mathbf{B})^{k_2+2} (\bar{\mathbf{u}}_{i\cdot}) \right)_{\alpha}^{\alpha}}{\sum_{\alpha \in I_{s_2, q}} \left| (\mathbf{W}_2 \mathbf{B})^{k_2+2} \right|_{\alpha}^{\alpha}}, \quad (49)$$

where $\bar{\mathbf{u}}_{i\cdot}$ is the i -th row of $\bar{\mathbf{U}} = \mathbf{B}(\mathbf{W}_2 \mathbf{B})^{k_2}$ for all $i = \overline{1, p}$.

By component-wise writing (41) we obtain,

$$x_{ij} = \sum_{s=1}^p \left(\sum_{t=1}^n a_{it}^{d, W_1} d_{ts} \right) \cdot b_{sj}^{d, W_2} \quad (50)$$

Denote by $\hat{\mathbf{d}}_{\cdot s}$ the s -th column of $\bar{\mathbf{V}}\mathbf{D} = (\mathbf{A}\mathbf{W}_1)^{k_1}\mathbf{A}\mathbf{D} =: \hat{\mathbf{D}} = (\hat{d}_{ij}) \in \mathbb{H}^{m \times p}$ for all $s = \overline{1, p}$. It follows from $\sum_t \bar{\mathbf{v}}_{\cdot t} d_{ts} = \hat{\mathbf{d}}_{\cdot s}$ that

$$\begin{aligned} \sum_{t=1}^n a_{it}^{d, W_1} d_{ts} &= \sum_{t=1}^n \frac{\sum_{\beta \in J_{s_1, m}\{i\}} \text{cdet}_i \left((\mathbf{A}\mathbf{W}_1)^{k_1+2}_{\cdot i} (\bar{\mathbf{v}}_{\cdot t}) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{s_1, m}} \left| (\mathbf{A}\mathbf{W}_1)^{k_1+2}_{\cdot i} \right|_{\beta}^{\beta}} \cdot d_{ts} = \\ &= \frac{\sum_{\beta \in J_{s_1, m}\{i\}} \sum_{t=1}^n \text{cdet}_i \left((\mathbf{A}\mathbf{W}_1)^{k_1+2}_{\cdot i} (\bar{\mathbf{v}}_{\cdot t}) \right)_{\beta}^{\beta} \cdot d_{ts}}{\sum_{\beta \in J_{s_1, m}} \left| (\mathbf{A}\mathbf{W}_1)^{k_1+2}_{\cdot i} \right|_{\beta}^{\beta}} = \\ &= \frac{\sum_{\beta \in J_{s_1, m}\{i\}} \text{cdet}_i \left((\mathbf{A}\mathbf{W}_1)^{k_1+2}_{\cdot i} (\hat{\mathbf{d}}_{\cdot s}) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{s_1, m}} \left| (\mathbf{A}\mathbf{W}_1)^{k_1+2}_{\cdot i} \right|_{\beta}^{\beta}} \quad (51) \end{aligned}$$

Suppose $\mathbf{e}_{s\cdot}$ and $\mathbf{e}_{\cdot s}$ are respectively the unit row-vector and the unit column-vector whose components are 0, except the s -th components, which are 1. Substituting (51) and (49) in (50), we obtain

$$x_{ij} = \sum_{s=1}^p \frac{\sum_{\beta \in J_{s_1, m}\{i\}} \text{cdet}_i \left((\mathbf{A}\mathbf{W}_1)^{k_1+2}_{\cdot i} (\hat{\mathbf{d}}_{\cdot s}) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{s_1, m}} \left| (\mathbf{A}\mathbf{W}_1)^{k_1+2}_{\cdot i} \right|_{\beta}^{\beta}} \frac{\sum_{\alpha \in I_{s_2, q}\{j\}} \text{rdet}_j \left((\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\cdot} (\bar{\mathbf{u}}_{s\cdot}) \right)_{\alpha}^{\alpha}}{\sum_{\alpha \in I_{s_2, q}} \left| (\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\cdot} \right|_{\alpha}^{\alpha}}.$$

Since

$$\hat{\mathbf{d}}_{\cdot s} = \sum_{t=1}^n \mathbf{e}_{\cdot t} d_{ts}, \quad \bar{\mathbf{u}}_{s\cdot} = \sum_{l=1}^q \bar{u}_{sl} \mathbf{e}_{l\cdot}, \quad \sum_{s=1}^p d_{ts} \bar{u}_{sl} = \bar{d}_{tl}, \quad (52)$$

then we have

$$\begin{aligned} x_{ij} &= \frac{\sum_{s=1}^p \sum_{t=1}^n \sum_{l=1}^q \sum_{\beta \in J_{s_1, m}\{i\}} \text{cdet}_i \left((\mathbf{A}\mathbf{W}_1)^{k_1+2}_{\cdot i} (\mathbf{e}_{\cdot t}) \right)_{\beta}^{\beta} \hat{d}_{ts} \bar{u}_{sl} \sum_{\alpha \in I_{s_2, q}\{j\}} \text{rdet}_j \left((\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\cdot} (\mathbf{e}_{l\cdot}) \right)_{\alpha}^{\alpha}}{\sum_{\beta \in J_{s_1, m}} \left| (\mathbf{A}\mathbf{W}_1)^{k_1+2}_{\cdot i} \right|_{\beta}^{\beta} \sum_{\alpha \in I_{s_2, q}} \left| (\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\cdot} \right|_{\alpha}^{\alpha}} = \\ &= \frac{\sum_{t=1}^n \sum_{l=1}^q \sum_{\beta \in J_{s_1, m}\{i\}} \text{cdet}_i \left((\mathbf{A}\mathbf{W}_1)^{k_1+2}_{\cdot i} (\mathbf{e}_{\cdot t}) \right)_{\beta}^{\beta} \bar{d}_{tl} \sum_{\alpha \in I_{s_2, q}\{j\}} \text{rdet}_j \left((\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\cdot} (\mathbf{e}_{l\cdot}) \right)_{\alpha}^{\alpha}}{\sum_{\beta \in J_{s_1, m}} \left| (\mathbf{A}\mathbf{W}_1)^{k_1+2}_{\cdot i} \right|_{\beta}^{\beta} \sum_{\alpha \in I_{s_2, q}} \left| (\mathbf{W}_2\mathbf{B})^{k_2+2}_{j\cdot} \right|_{\alpha}^{\alpha}}. \quad (53) \end{aligned}$$

Denote by

$$d_{il}^{\mathbf{A}} := \sum_{\beta \in J_{s_1, m} \{i\}} \text{cdet}_i \left((\mathbf{A}\mathbf{W}_1)_{\cdot i}^{k_1+2} (\bar{\mathbf{d}}_{\cdot l}) \right)_{\beta}^{\beta} = \sum_{t=1}^n \sum_{\beta \in J_{s_1, m} \{i\}} \text{cdet}_i \left((\mathbf{A}\mathbf{W}_1)_{\cdot i}^{k_1+2} (\mathbf{e}_{\cdot t}) \right)_{\beta}^{\beta} \bar{d}_{tl}$$

the l -th component of a row-vector $\mathbf{d}_{i\cdot}^{\mathbf{A}} = (d_{i1}^{\mathbf{A}}, \dots, d_{iq}^{\mathbf{A}})$ for all $l = \overline{1, q}$. Substituting it in (53), we have

$$x_{ij} = \frac{\sum_{l=1}^q d_{il}^{\mathbf{A}} \sum_{\alpha \in I_{s_2, q} \{j\}} \text{rdet}_j \left((\mathbf{W}_2 \mathbf{B})_{j\cdot}^{k_2+2} (\mathbf{e}_{l\cdot}) \right)_{\alpha}^{\alpha}}{\sum_{\beta \in J_{s_1, m}} \left| (\mathbf{A}\mathbf{W}_1)^{k_1+2} \right|_{\beta}^{\beta} \sum_{\alpha \in I_{s_2, q}} \left| (\mathbf{W}_2 \mathbf{B})^{k_2+2} \right|_{\alpha}^{\alpha}}.$$

Since $\sum_{l=1}^q d_{il}^{\mathbf{A}} \mathbf{e}_{l\cdot} = \mathbf{d}_{i\cdot}^{\mathbf{A}}$, then it follows (44).

If we denote by

$$d_{tj}^{\mathbf{B}} := \sum_{l=1}^q \bar{d}_{tl} \sum_{\alpha \in I_{s_2, q} \{j\}} \text{rdet}_j \left((\mathbf{W}_2 \mathbf{B})_{j\cdot}^{k_2+2} (\mathbf{e}_{l\cdot}) \right)_{\alpha}^{\alpha} = \sum_{\alpha \in I_{s_2, q} \{j\}} \text{rdet}_j \left((\mathbf{W}_2 \mathbf{B})_{j\cdot}^{k_2+2} (\bar{\mathbf{d}}_{t\cdot}) \right)_{\alpha}^{\alpha} \quad (54)$$

the t -th component of a column-vector $\mathbf{d}_{\cdot j}^{\mathbf{B}} = (d_{1j}^{\mathbf{B}}, \dots, d_{nj}^{\mathbf{B}})^T$ for all $t = \overline{1, n}$ and substituting it in (53), we obtain

$$x_{ij} = \frac{\sum_{t=1}^n \sum_{\beta \in J_{s_1, m} \{i\}} \text{cdet}_i \left((\mathbf{A}\mathbf{W}_1)_{\cdot i}^{k_1+2} (\mathbf{e}_{\cdot t}) \right)_{\beta}^{\beta} d_{tj}^{\mathbf{B}}}{\sum_{\beta \in J_{r_1, n}} \left| (\mathbf{A}^* \mathbf{A}) \right|_{\beta}^{\beta} \sum_{\alpha \in I_{r_2, p}} \left| (\mathbf{B}\mathbf{B}^*) \right|_{\alpha}^{\alpha}}.$$

Since $\sum_{t=1}^n \mathbf{e}_{\cdot t} d_{tj}^{\mathbf{B}} = \mathbf{d}_{\cdot j}^{\mathbf{B}}$, then it follows (43). \square

4 Examples

In this section, we give examples to illustrate our results.

1. Let us consider the matrix equation

$$\mathbf{W}\mathbf{A}\mathbf{W}\mathbf{X} = \mathbf{D} \quad (55)$$

with the restricted conditions (29), where

$$\mathbf{A} = \begin{pmatrix} 0 & i & 0 \\ k & 1 & i \\ 1 & 0 & 0 \\ 1 & -k & -j \end{pmatrix}, \quad \mathbf{W} = \begin{pmatrix} k & 0 & i & 0 \\ -j & k & 0 & 1 \\ 0 & 1 & 0 & -k \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} k & i \\ i & -j \\ 1 & -i \end{pmatrix}.$$

Then

$$\mathbf{V} = \mathbf{A}\mathbf{W} = \begin{pmatrix} -k & -j & 0 & i \\ -1-j & i+k & j & 1+j \\ k & 0 & i & 0 \\ -i+k & 1-j & i & i-k \end{pmatrix}, \quad \mathbf{U} = \mathbf{W}\mathbf{A} = \begin{pmatrix} i & j & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $\text{rank } \mathbf{W} = 3$, $\text{rank } \mathbf{V} = 3$, $\text{rank } \mathbf{V}^3 = \text{rank } \mathbf{V}^2 = 2$, $\text{rank } \mathbf{U}^2 = \text{rank } \mathbf{U} = 2$. So, $\text{Ind } \mathbf{V} = 2$, $\text{Ind } \mathbf{U} = 1$, and $k = \max\{\text{Ind}(\mathbf{A}\mathbf{W}), \text{Ind}(\mathbf{W}\mathbf{A})\} = 2$.

We shall find the W -weighted Drazin inverse solution of (55) by its determinantal representation (31). We have

$$\begin{aligned} \mathbf{U}^2 &= \begin{pmatrix} -1 & i+k & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{U}^5 = \begin{pmatrix} i & 2+3j & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ (\mathbf{U}^5)^* &= \begin{pmatrix} -i & 0 & 0 \\ 2-3j & -k & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\mathbf{U}^5)^* \mathbf{U}^5 = \begin{pmatrix} 1 & -2i-3k & 0 \\ 2i+3k & 14 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \hat{\mathbf{D}} &= (\mathbf{U}^5)^* \mathbf{U}^2 \mathbf{D} = \begin{pmatrix} i-j-k & -j \\ 1+3i+6j-2k & 4i-2k \\ 0 & 0 \end{pmatrix}, \quad \mathbf{W}^* = \begin{pmatrix} -k & j & 0 \\ 0 & -k & 1 \\ -i & 0 & 0 \\ 0 & 1 & k \end{pmatrix}, \\ \mathbf{W}^* \mathbf{W} &= \begin{pmatrix} 2 & i & -j & j \\ -i & 2 & 0 & -2k \\ j & 0 & 1 & 0 \\ -j & 2k & 0 & 2 \end{pmatrix}, \quad \hat{\mathbf{W}} = \mathbf{W}^* \mathbf{U}^2 = \begin{pmatrix} -k & 1-2j & 0 \\ 0 & i+k & 0 \\ i & 1+j & 0 \\ 0 & -1 & 0 \end{pmatrix}. \end{aligned}$$

Since by (31)

$$x_{11} = \frac{\sum_{t=1}^3 \sum_{\beta \in I_{3,4}\{1\}} \text{cdet}_1((\mathbf{W}^* \mathbf{W})_{.1}(\hat{\mathbf{w}}_{.t}))_{\beta}^{\beta} \sum_{\beta \in J_{2,3}\{t\}} \text{cdet}_t\left(\left((\mathbf{U}^5)^* \mathbf{U}^5\right)_{.t}(\hat{\mathbf{d}}_{.1})\right)_{\beta}^{\beta}}{\sum_{\beta \in J_{3,4}} \left|(\mathbf{W}^* \mathbf{W})_{\beta}^{\beta}\right| \sum_{\beta \in J_{2,3}} \left|((\mathbf{U}^5)^* \mathbf{U}^5)_{\beta}^{\beta}\right|},$$

where

$$\begin{aligned} \sum_{\beta \in I_{3,4}\{1\}} \text{cdet}_1((\mathbf{W}^* \mathbf{W})_{.1}(\hat{\mathbf{w}}_{.1}))_{\beta}^{\beta} &= \\ \text{cdet}_1 \begin{pmatrix} k & i & -j \\ 0 & 2 & 0 \\ i & 0 & 1 \end{pmatrix} &+ \text{cdet}_1 \begin{pmatrix} k & i & j \\ 0 & 2 & -2k \\ 0 & 2k & 1 \end{pmatrix} + \text{cdet}_1 \begin{pmatrix} k & -j & j \\ i & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 0, \end{aligned}$$

$$\sum_{\beta \in I_{3,4}\{3\}} \text{cdet}_1((\mathbf{W}^* \mathbf{W})_{.1}(\hat{\mathbf{w}}_{.2}))_{\beta}^{\beta} = -2j,$$

$$\sum_{\beta \in I_{3,4}\{1\}} \text{cdet}_1((\mathbf{W}^* \mathbf{W})_{.1}(\hat{\mathbf{w}}_{.3}))_{\beta}^{\beta} = 0, \quad \sum_{\beta \in J_{3,4}} \left|(\mathbf{W}^* \mathbf{W})_{\beta}^{\beta}\right| = 2,$$

and

$$\begin{aligned}
& \sum_{\beta \in J_{2,3}\{1\}} \text{cdet}_1 \left(\left((\mathbf{U}^5)^* \mathbf{U}^5 \right)_{\cdot 1} (\hat{\mathbf{d}}_{\cdot 1}) \right)_{\beta}^{\beta} = \\
& \text{cdet}_1 \begin{pmatrix} i-j-k & -2i-3k \\ 1+3i+6j-2k & 14 \end{pmatrix} + \text{cdet}_1 \begin{pmatrix} i-j-k & 0 \\ 0 & 0 \end{pmatrix} = -2i-j-k, \\
& \sum_{\beta \in J_{2,3}\{2\}} \text{cdet}_2 \left(\left((\mathbf{U}^5)^* \mathbf{U}^5 \right)_{\cdot 2} (\hat{\mathbf{d}}_{\cdot 1}) \right)_{\beta}^{\beta} = j, \\
& \sum_{\beta \in J_{2,3}\{3\}} \text{cdet}_3 \left(\left((\mathbf{U}^5)^* \mathbf{U}^5 \right)_{\cdot 3} (\hat{\mathbf{d}}_{\cdot 1}) \right)_{\beta}^{\beta} = 0, \quad \sum_{\beta \in J_{2,3}} \left| \left((\mathbf{U}^5)^* \mathbf{U}^5 \right)_{\beta}^{\beta} \right| = 1,
\end{aligned}$$

then

$$\begin{aligned}
x_{11} &= \frac{0 \cdot (-2i-j-k) + (-2j) \cdot j + 0 \cdot 0}{2 \cdot 1} = 1, \\
x_{12} &= \frac{0 \cdot (-2+2j) + (-2j) \cdot i + 0 \cdot 0}{2 \cdot 1} = k, \\
x_{21} &= \frac{2j \cdot (-2i-j-k) + (10i-4k) \cdot j + 0 \cdot 0}{2 \cdot 1} = 1+i+7k, \\
x_{22} &= \frac{2j \cdot (-2+2j) + (10i-4k) \cdot i + 0 \cdot 0}{2 \cdot 1} = -7-4j, \\
x_{31} &= \frac{10i \cdot (-2i-j-k) + j \cdot j + 0 \cdot 0}{2 \cdot 1} = 9.5+5j-5k, \\
x_{32} &= \frac{10i \cdot (-2+2j) + j \cdot i + 0 \cdot 0}{2 \cdot 1} = -10i+9.5k,
\end{aligned}$$

We finally get,

$$\mathbf{X} = \begin{pmatrix} 1 & k \\ 1+i+7k & -7-4j \\ 9.5+5j-5k & -10i+9.5k \end{pmatrix}. \quad (56)$$

2. Let now we consider the matrix equation

$$\mathbf{W}_1 \mathbf{A} \mathbf{W}_1 \mathbf{X} \mathbf{W}_2 \mathbf{B} \mathbf{W}_2 = \mathbf{D}, \quad (57)$$

with the constraints (3.1), where

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} k & 0 & i & 0 \\ -j & k & 0 & 1 \\ 0 & 1 & 0 & -k \end{pmatrix}, \quad \mathbf{W}_1 = \begin{pmatrix} k & -j & 0 \\ 0 & k & 1 \\ i & 0 & 0 \\ 0 & 1 & -k \end{pmatrix}, \quad \mathbf{W}_2 = \begin{pmatrix} k & -i \\ j & 0 \\ 0 & 1 \end{pmatrix}, \\
\mathbf{B} &= \begin{pmatrix} k & j & 0 \\ j & 0 & 1 \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} i & -1 \\ k & 0 \\ 0 & j \\ -1 & 0 \end{pmatrix}.
\end{aligned}$$

Since the following matrices are Hermitian

$$\mathbf{V} = \mathbf{A} \mathbf{W}_1 = \begin{pmatrix} -2 & i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{U} = \mathbf{W}_2 \mathbf{B} = \begin{pmatrix} 0 & -i & -i \\ i & -1 & 0 \\ i & 0 & -1 \end{pmatrix},$$

then we can find the W -weighted Drazin inverse solution of (57) by its determinantal representation (43).

We have

$$\begin{aligned} k_1 &= \max \{ \text{Ind}(\mathbf{A}\mathbf{W}_1), \text{Ind}(\mathbf{W}_1\mathbf{A}) \} = 1, \\ k_2 &= \max \{ \text{Ind}(\mathbf{B}\mathbf{W}_2), \text{Ind}(\mathbf{W}_2\mathbf{B}) \} = 1, \end{aligned}$$

and $s_1 = \text{rank}(\mathbf{A}\mathbf{W}_1) = 2$, $s_2 = \text{rank}(\mathbf{W}_2\mathbf{B}) = 2$. Since

$$(\mathbf{A}\mathbf{W}_1)^3 = \begin{pmatrix} -13 & 8i & 0 \\ -8i & -5 & 0 \\ 0 & 0 & 0 \end{pmatrix}, (\mathbf{W}_2\mathbf{B})^3 = \begin{pmatrix} 0 & -3i & -3i \\ 3i & -3 & 0 \\ 3i & 0 & 3 \end{pmatrix},$$

then

$$\sum_{\beta \in J_{2,3}} |(\mathbf{A}\mathbf{W}_1)^3{}^\beta{}_\beta| = 1, \sum_{\alpha \in I_{2,3}} |(\mathbf{W}_2\mathbf{B})^3{}^\alpha{}_\alpha| = -27.$$

We have

$$\bar{\mathbf{D}} = \mathbf{A}\mathbf{W}_1\mathbf{A}\mathbf{D}\mathbf{B}\mathbf{W}_2\mathbf{B} = \begin{pmatrix} 2i+j & -7+k & -5+2k \\ -1+k & -5i-j & -4i-2j \\ 0 & 0 & 0 \end{pmatrix},$$

By (45), we can get

$$\mathbf{d}_{\cdot 1}^{\mathbf{B}} = \begin{pmatrix} 36i-9j \\ -27-9k \\ 0 \end{pmatrix}, \quad \mathbf{d}_{\cdot 2}^{\mathbf{B}} = \begin{pmatrix} -27 \\ -18i \\ 0 \end{pmatrix}, \quad \mathbf{d}_{\cdot 3}^{\mathbf{B}} = \begin{pmatrix} 9-9k \\ 9i+3j \\ 0 \end{pmatrix}.$$

Since

$$(\mathbf{A}\mathbf{W}_1)^3{}_{\cdot 1}(\mathbf{d}_{\cdot 1}^{\mathbf{B}}) = \begin{pmatrix} 36i-9j & 8i & 0 \\ -27-9k & -5 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

then finally we obtain

$$x_{11} = \frac{\sum_{\beta \in J_{2,3}\{1\}} \text{cdet}_1 \left((\mathbf{A}\mathbf{W}_1)^3{}_{\cdot 1}(\mathbf{d}_{\cdot 1}^{\mathbf{B}}) \right)^\beta{}_\beta}{\sum_{\beta \in J_{2,3}} |(\mathbf{A}\mathbf{W}_1)^3{}^\beta{}_\beta| \sum_{\alpha \in I_{2,3}} |(\mathbf{W}_2\mathbf{B})^3{}^\alpha{}_\alpha|} = \frac{36i-27j}{-27} = \frac{-4i+3j}{3},$$

Similarly,

$$\begin{aligned} x_{12} &= \frac{\text{cdet}_1 \begin{pmatrix} -27 & 8i \\ -18i & -5 \end{pmatrix}}{-27} = \frac{1}{3}, \quad x_{13} = \frac{\text{cdet}_1 \begin{pmatrix} 9-9k & 8i \\ 9i-3j & -5 \end{pmatrix}}{-27} = \frac{-9-7k}{9}, \\ x_{21} &= \frac{\text{cdet}_2 \begin{pmatrix} -13 & 36i-9j \\ -8i & -27-9k \end{pmatrix}}{-27} = \frac{-7-5k}{3}, \quad x_{22} = \frac{\text{cdet}_2 \begin{pmatrix} -13 & -27 \\ -8i & -18i \end{pmatrix}}{-27} = \frac{-2i}{3}, \\ x_{23} &= \frac{\text{cdet}_2 \begin{pmatrix} -13 & -9-9k \\ -8i & 9i+3j \end{pmatrix}}{-27} = \frac{15i-11j}{9}, \quad x_{31} = x_{32} = x_{33} = 0. \end{aligned}$$

So, the W-weighted Drazin inverse solution of (57) are

$$\mathbf{X} = \frac{1}{9} \begin{pmatrix} -12i + 9j & 3 & -9 - 7k \\ -21 - 15k & -6i & 15i - 11j \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that we used Maple with the package CLIFFORD in the calculations.

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